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On the edge double Roman domination number of graphs

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Abstract

An *edge double Roman dominating function* (EDRDF) on a graph G is a function $f : E(G) \to \{0, 1, 2, 3\}$ satisfying the condition that such that every edge e with $f(e) = 0$, is adjacent to at least two edge e, e' for which $f(e) = f(e') = 2$ or one edge e'' with $f(e'') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geqslant 2$. The Edge double Roman dominating number of G, denoted by $\gamma'_{dR}(G)$, is the minimum weight $w(f) = \sum_{e \in E(G)} f(e)$ of an edge double Roman dominating function f of G. In this paper, we introduction some results on the edge double Roman domination number of a graph. Also, we provide some upper and lower bounds for the edge double Roman domination number of graphs.

Keywords: Double Roman dominating function, Double Roman domination number, Edge double Roman dominating function, Edge double Roman domination number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A graph G is k-*regular* if $d(v) = k$ for each vertex v of G. A *leaf* is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf, and a *strong support vertex* is a support vertex adjacent to at least two leaves. An edge incident to a leaf is called a *pendant edge*. A *tree* is an acyclic connected graph. A tree T is a *double star* if it contains exactly two vertices that are not leaves. For $a, b \geqslant 2$, a double star whose support vertices have degree a and b is denoted by $S(a, b)$. If T is a rooted tree, we for each vertex v, we denote by T_v the sub-rooted tree rooted at v. The height of a rooted is the maximum distance from the root to a leaf.

The complement of a graph G is denote by $\overline{G}$. We write K_n for the complete graph of order n, C_n for the cycle of length n, and P_n for the path of order n. A *matching* is any independent set of edges. A maximal matching is a matching X so that $V(X) - V(G)$ is an independent set of vertices. A *perfect matching* in graph G is a matching so that $V(X) = V(G)$. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ee' \in E(L(G))$ when $e = uv$ and $e' = vw$ in G. It is easy to see that $L(K_{1,n}) = K_n, L(C_n) = C_n$ and $L(P_n) = P_{n-1}$. For a subset S of vertices of G, and a vertex $x \in S$,

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we may that a vertex $y \notin S$ is a *private neighbor* of x with respect to S if $N(y) \cap S = \{x\}$.
A *double Roman dominating function* on a graph G is defined by Beeler, Haynes and Hedetniemi in [7] as a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(u) = 0$, then vertex u has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(u) = 1$, then vertex u must have at least one neighbor w with $f(w) \geqslant 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G))$. The *double Roman domination number* of a graph G, denoted by $\gamma_{dR}(G)$, is the minimum weight of any double Roman dominating function of G. Further results on the double Roman domination number can be found in [5, 3, 1].
A subset X of $E(G)$ is called an *edge dominating set* of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. We refer to an edge dominating set with minimum cardinality as a $\gamma'(G)$-set. The concept of edge domination was introduced by Mitchell and Hedetniemi[13], and further studied for example in [12, 6].
A *Edge Roman dominating function*(ERDF) of graph G is a function $f : E(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$. The *Edge Roman domination number* of a graph G, denoted by $\gamma'_R(G)$, is the minimum weight $w(f) = \sum_{e\in E(G)} f(e)$ of an Edge Roman dominating function of G. The concept of edge Roman domination has been several variants of domination, see for example [9, 10, 14, 15, 8, 4]
A *Edge double Roman dominating function*(EDRDF) of graph G is a function $f : E(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(e) = 0$, then edge e has at least two neighbors assigned 2 under f or one neighbor e' with $f(e') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geqslant 2$. The *weight* of an edge double Roman dominating number of f, denote by $\omega(f)$, is the value $\sum_{e\in E(G)} f(e)$. The weight of a EDRDF, $\sum_{e\in E(G)} f(e)$. The minimum weight of a $EDRDF$ is the edge double roman domination number of G, denoted by $\gamma'_{dR}(G)$. If f is a $EDRDF$ in a graph G, then we simply can represent f by $f = (E_0, E_1, E_2, E_3)$(or $f = (E_0^f, E_1^f, E_2^f, E_3^f)$ to refer to f), where $E_0 = \{e \in E(G) : f(e) = 0\}, E_1 = \{e \in E(G) : f(e) = 1\}, E_2 = \{e \in E(G) : f(e) = 2\}$, and $E_3 = \{e \in E(G) : f(e) = 3\}$.
In this note we initiate the study of the Edge double Roman domination in graphs and present some (sharp) bounds for this parameter. In addition, we determine the Edge double Roman domination number of some classes of graphs.

2. Preliminaries and Exact Values

We begin by showing that for any graph G, there exists a γ'_{dR}-function of G where no edge is assigned a 1, that is, $E_1 = \emptyset$ for some γ'_{dR}-function of G.

Proposition 2.1. *In a double Roman dominating function of weight $\gamma'_{dR}(G)$, no edge needs to be assigned the value 1.*

Proof. Let f be a γ'_{dR}-function on a graph G. Suppose that for some $e \in E$, $f(e) = 1$. This means that there is a edge $e' \in N(e)$, such that either $f(e') = 2$ or $f(e') = 3$. If $f(e') = 3$, then we can achieve a Edge double Roman dominating function by reassigning a 0 to e. This results in a function with strictly less weight than f, contradicting that f is a γ'_{dR}-function of G. If $f(e') = 2$, then we can create a Edge double Roman domination function g defined as follows: $g(x) = f(x)$ for all $x \notin \{e, e'\}, g(e) = 0$, and $g(e') = 3$. This result in a double Roman domination function with weight equal to f. □

Let $\mathcal{H}$ be the family of connected graphs G of order n that can be built from $\frac{n}{4}$copies of P_4 by adding a connected subgraph on the set of centers of $\frac{n}{4}P_4$. We make use of the following.

Theorem A. *[7] If G is a connected graph of order $n \geqslant 3$, then $\gamma_{dR}(G) \leqslant \frac{5n}{4}$, with equality if and only if $G \in \mathcal{H}$.*

Theorem B. *If $n \geqslant 1$, then $\gamma_{dR}(P_n) = n$ for $n \equiv 0 \pmod 3$ and $\gamma_{dR}(P_n) = n + 1$ for $n \equiv 1 \text{ or } 2 \pmod 3$*

Theorem C. *If* $n \geqslant 3$*, then* $\gamma_{dR}(C_n) = n$ *for* $n \equiv 0\ ,2\ ,3\ ,4 \pmod 6$ *and* $\gamma_{dR}(C_n) = n+1$ *for* $n \equiv 1\,,5 \pmod 6$

Theorem D. *For* $n \geqslant 2$*,* $\gamma_{dR}(K_n) = 3$

The following is obvious.

observation 2.2. *For any nonempty graph* G *of order* $n \geqslant 2$*,*

$$\gamma'_{dR}(G) = \gamma_{dR}(L(G))$$

Corollary 2.3. *For* $n \geqslant 2$*,* $\gamma'_{dR}(K_{1,n}) = 3$.

Corollary 2.4. *If* $n \geqslant 1$*, then* $\gamma'_{dR}(P_n) = n$ *for* $n \equiv 0 \pmod 3$ *and* $\gamma'_{dR}(P_n) = n-1$ *for* $n \equiv 1 \,\text{or}\, 2 \pmod 3$

Corollary 2.5. *If* $n \geqslant 3$*, then* $\gamma'_{dR}(C_n) = n$ *for* $n \equiv 0\ ,2\ ,3\ ,4 \pmod 6$ *and* $\gamma'_{dR}(C_n) = n+1$ *for* $n \equiv 1\,,5 \pmod 6$

Proposition 2.6. *For a complete graph* K_n *with* $n \geqslant 4$*,* $\gamma'_{dR}(K_n) = n$ *if* n *is even, and* $\gamma'_{dR}(K_n) = n+1$ *if* n *is odd.*

Proof. First let $n \geqslant 4$ be even. Let $\{e_1, e_2, ..., e_{\frac{n}{2}}\}$ be a perfect matching of K_n. Then assigning 2 to each e_i, $i > 1$, and 0 to each other edge produces a EDRDF for K_n, thus $\gamma'_{dR}(K_n) \leqslant n$. We use on induction on n to show that $\gamma'_{dR}(K_n) \geqslant n$. The basis step of the induction is obvious for $n = 4$. Assume the result holds for any even integer $n' < n$. Assume that $\gamma'_{dR}(K_n) \leqslant n-1$. Let f be a $\gamma'_{dR}(K_n)$-function. Let there is edge xx_1 in G such that $f(xx_1) = 3$. Let $G = K_n - \{x, x_1\}$, clearly $G \equiv K_{n-2}$, and $f|_V(G)$ is EDRDF for G. By the induction hypothesis $w(f|_V(G)) \geqslant \gamma'_{dR}(K_{n-2}) \geqslant n-2$. Consequently, $\gamma'_{dR}(K_{n-2}) \leqslant w(f|_{K_{n-2}}) = w(f) - 3 = \gamma'_{dR}(K_n) - 3 \leqslant n-4$, a contradiction. Thus $\gamma'_{dR}(K_n) \geqslant n$.
Now assume there is edge xx_1 in G such that $f(xx_1) = 2$. Let $G = K_n - \{x, x_1\}$, clearly $G \equiv K_{n-2}$, and $f|_V(G)$ is EDRDF for G. By the induction hypothesis $w(f|_V(G)) \geqslant \gamma'_{dR}(K_{n-2}) \geqslant n-2$. Consequently, $\gamma'_{dR}(K_{n-2}) \leqslant w(f|_{K_{n-2}}) = w(f) - 2 = \gamma'_{dR}(K_n) - 2 \leqslant n-3$, a contradiction. Thus $\gamma'_{dR}(K_n) \geqslant n$.
Now, let $n \geqslant 3$ be odd. Let $\{e_1, e_2, ..., e_{\frac{n-1}{2}}\}$ be a minimum matching of K_n. Then assigning 2 to each e_i, $i = 1, 2, ..., \frac{n-1}{2}$, and $f(e_n) = 2$, and 0 to each other edge produces a EDRDF for K_n, thus $\gamma'_{dR}(K_n) \leqslant n+1$. We use on induction on n to show that $\gamma'_{dR}(K_n) \geqslant n+1$. The basis step of the induction is obvious for $n = 3$. Assume the result holds for any odd integer $5 \leqslant n' < n$. Assume that $\gamma'_{dR}(K_n) \leqslant n$. Let f be a $\gamma'_{dR}(K_n)$-function. Let there is edge xx_1 in G such that $f(xx_1) = 3$. Let $G = K_n - \{x, x_1\}$, clearly $G \equiv K_{n-2}$, and $f|_V(G)$ is EDRDF for G. By the induction hypothesis $w(f|_V(G)) \geqslant \gamma'_{dR}(K_{n-2}) \geqslant n-1$. Consequently, $\gamma'_{dR}(K_{n-2}) \leqslant w(f|_{K_{n-2}}) = w(f) - 3 = \gamma'_{dR}(K_n) - 3 \leqslant n-3$, a contradiction. Thus $\gamma'_{dR}(K_n) \geqslant n+1$.
Now assume there is edge xx_1 in G such that $f(xx_1) = 2$. Let $G = K_n - \{x, x_1\}$, clearly $G \equiv K_{n-2}$, and $f|_V(G)$ is EDRDF for G. By the induction hypothesis $w(f|_V(G)) \geqslant \gamma'_{dR}(K_{n-2}) \geqslant n-1$. Consequently, $\gamma'_{dR}(K_{n-2}) \leqslant w(f|_{K_{n-2}}) = w(f) - 2 = \gamma'_{dR}(K_n) - 2 \leqslant n-2$, a contradiction. Thus $\gamma'_{dR}(K_n) \geqslant n+1$. □

Proposition 2.7. *Let* $K_{r,s}$ *be a complete bipartite graph with partite sets* $X = \{x_1, x_2, ..., x_r\}$ *and* $Y = \{y_1, y_2, ..., y_s\}$*. If* $1 \leqslant r \leqslant s$ *and* $s = r + i$*, then*

$$\gamma'_{dR}(K_{r,s}) = \begin{cases} 2s & \text{if} \quad r > 2i \\ 3r & \text{if} \quad r \leqslant 2i \end{cases}$$

Proof. For $r \geqslant 2i$, the function f defined by $f(x_i y_i) = 3$ for $1 \leqslant i \leqslant r$ and $f(x_i y_j) = 0$ for $i \neq j$, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ is an edge double Roman dominating function of weight 3r, which gives $\gamma'_{dR}(K_{r,s} \leqslant 3r$. Now proof that $\gamma'_{dR}(K_{r,s}) \geqslant 3r$.

suppose f is an edge double Roman dominating function of $K_{r,s}$ with the minimum weight. Assume $a = |\{e \in E(G) : f(e) = 3\}|$ and $b = |\{e \in E(G) : f(e) = 2\}|$. If $a \geqslant r$ and $r \leqslant 2i$, then $\gamma'_{dR}(K_{r,s}) = w(f) \geqslant 3r$. If $a < r$, $r \leqslant 2i$ and $b \geqslant s$, then $\gamma'_{dR}(K_{r,s}) \geqslant 2s = 2(r+i) = 2r + 2i \geqslant 3r$. If $a < r$ and $b < r$. Let X contain at least a vertex u such that $N(u) \cap E_3 = \emptyset$ and let Y contain at least a vertex v such that $N(v) \cap E_2 = \emptyset$. Since G is complete bipartite graph $uv \in E(G)$ and uv not incident to any edge e with $f(e) = 3$ or $f(e) = 2$, a contradiction. Thus $a \geqslant r$ or $b \geqslant s$ and we are done.

For $r > 2i$, the function f defined by $f(x_i y_i) = 2$ for $1 \leqslant i \leqslant r$ and $f(x_r y_j) = 2$ for $i \neq j$, $1 \leqslant j \leqslant s$ is an edge double Roman dominating function of weight 2s, which gives $\gamma'_{dR}(K_{r,s} \leqslant 2s$. Now proof that $\gamma'_{dR}(K_{r,s}) \geqslant 2s$. If $b \geqslant s$,then $\gamma'_{dR}(K_{r,s}) \geqslant 2b \geqslant 2s$. If $b < s$ and $a \geqslant r$, then $\gamma'_{dR}(K_{r,s}) \geqslant 3a \geqslant 3r = 3(s-i) = 4s-3i = 2s+s-3i = 2s+r+i-3i = 2s+r-2i > 2s$, a contradiction. If $b < s$ and $a < r$,as above, a contradiction. □

3. Bounds

Proposition 3.1. *For any connected graph* G, $2\gamma'(G) \leqslant \gamma'_{dR}(G) \leqslant 3\gamma'(G)$. *Equality for the upper bound holds if and only if there is a* $\gamma'_{dR}(G)$*-function with* $E_2 = \varnothing$. *Equality for the lower bound holds if and only if* $G \in \{K_2, C_4, K_4\}$.

Proof. The bounds are obvious we prove the equality parts. For the upper bound, let G be a connected graph with $\gamma'_{dR}(G) = 3\gamma'(G)$. Let S be a $\gamma'(G)$-set. Then assigning 3 to each edge of S, and 0 to each other edge of G, produces a desired $\gamma'_{dR}(G)$- function. Conversely, assume that a $\gamma'_{dR}(G)$-function f with $E_2 = \varnothing$. Then E_3 is an edge dominating set for G, thus $\gamma'(G) \leqslant \frac{\gamma'_{dR}(G)}{3}$. Now, the result follows. Next, we consider the equality of the lower bound let $f = (E_0, E_2, E_3)$ be any $\gamma'_{dR}(G)$-function. It is well known that $E_2 \cup E_3$ is an edge dominating set for G. Hence $\gamma'(G) \leqslant |E_2| + |E_3|$. Thus,

$$\gamma'_{dR}(G) = 2|E_2| + 3|E_3| \geqslant 2(|E_2| + |E_3|) \geqslant 2\gamma'(G)$$

It follows that $|E_3| = 0$. Suppose that E_2 is not independent. Let $\{e_1, e_2\} \in E_2$ such that e_1 and e_2 are incident at one common vertex. Then replacing $f(e_1)$ by 3, and $f(e_2)$ by 0 produces a $\gamma'_{dR}(G)$-function g with $|E_3^g| \neq 0$, a contradiction. Thus E_2 is independent.
Now, we show that $|E_2| \leqslant 2$. Suppose to the contrary, that $|E_2| \geqslant 3$. Let $ww_1, u_1a, u_2b \in E_2$ such that $u_1 \in N(w_1)$ and $u_2 \in N(w_2)$. Clearly, $f(u_1w_1) = f(u_2w_2) = 0$. Then $E_2 \cup \{u_1w_1, u_2w_2\} - \{u_1a, u_2b, w_1w_2\}$ is an edge domination set of G, contradiction. We conclude that $|E_2| \leqslant 2$. If $|E_2| = 1$, then clearly, $G = K_2$. Thus, assume that $|E_2| = 2$, so $G \in \{C_4, K_4\}$. □

Let $\mathcal{E}$ be the class of all graphs G that can be obtained from $k \geqslant 1$ double stars $S_1 = S(r_1, s_1), S_2 = S(r_2, s_2), ..., S_k = S(r_k, s_k)$, where $r_i, s_i \geqslant 2$ for $i = 1, 2, ..., k$, as follows. Let A be the set of central vertices of all stars $S_1, S_2, ..., S_k$. Then G is obtained from $S_1, S_2, ..., S_k$ by adding some edges between vertices of A, or adding some new vertices and joining each new vertex to at least two vertices of A, Figure 1 shows a graph in $\mathcal{E}$.

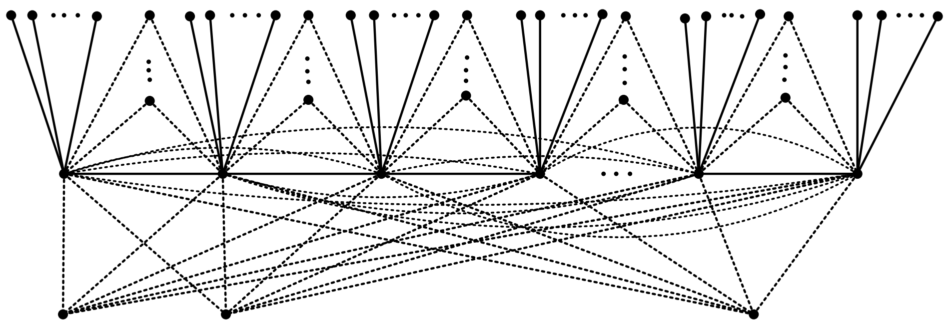

Figure 1: Structure of graphs in the family $\mathcal{E}$

Theorem 3.2. *For any connected graph* G *of order* $n \geqslant 2$, $\gamma_{dR}(G) \leqslant 2\gamma'_{dR}(G)$, *with equality if and only if* $G \in \mathcal{E}$.

Proof. Let G be a graph with no isolated vertex, and $f = (E_0, E_1, E_2, E_3)$ be any γ'_{dR}-function. Let g be defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_2 \cup E_3$ and 0 to any other vertex of G. Then g is a DRDF for G, and so $\gamma_{dR}(G) \leqslant 2\gamma'(G)$.

Assume that equality holds. let $f = (E_0, E_1, E_2, E_3)$ be a $\gamma'_{dR}(G)$-function such that $|E_3|$ is maximum. We show that E_3 is independent. Suppose that there are three u, v, w such that $uv \in E_3$ and $vw \in E_3$. Let g be defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_2 \cup E_3 - \{uv, vw\}$, 3 to u and w, and 0 to v and to any other vertex of G. Then g is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction. Thus E_3 is independent.

We next show that E_2 is independent. Suppose that there are three u, v, w such that $uv \in E_2$ and $vw \in E_2$. Let g be defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_2 \cup E_3 - \{uv, vw\}$, 2 to u and w, and 0 to v and to any other vertex of G. Then g is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction. Thus E_2 is independent.

We next show that for every edge $e = uv \in E_0$. at least one of u and v is incident on an edge of E_3. Suppose that there is an edge $e = uv \in E_0$ such that neither u nor v is incident on an edge of E_3. Then there are vertices $a \in N(u) - \{v\}$ and $b \in N(v) - \{u\}$ such that $ua \in E_2$ and $vb \in E_2$. Let g be defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_2 \cup E_3 - \{uv, vb\}$, 2 to b, and 0 to v and to any other vertex of G. Then g is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction. Thus for every edge $e = uv \in E_0$, at least one of u and v is incident on an edge of E_3.

We next show that $E_2 = \emptyset$. Suppose that $E_2 \neq \emptyset$. Let $e = uv \in E_2$. Since G is connected, we may assume that $deg(u) \geqslant 2$. Let $a \in N(u) - \{v\}$, clearly, $xa \in E_0$. By the previous argument there is a vertex $b \in N(a) - \{u\}$ such that $ab \in E_3$. Let g be a defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_2 \cup E_3 - \{uv\}$ and 2 to v and 0 to any other vertex of G. Then g is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction.

Let $S = \bigcup_{xy \in E_3} \{x, y\}$. Clearly, S is a dominating set for G. We show that $V(G) - S$ is independent. Suppose that $V(G) - S$ is not independent. Let ab be an edge with $a, b \in V(G) - S$. thus $f(ab) = 0$. Since $E_2 = \emptyset$, there is a vertex $t \in N(a) - b$ such that $f(ta) = 3$ or $t' \in N(b) - a$ such that $f(t'a) = 3$. Without loss of generality assume that $t \in N(a) - b$ and $f(ta) = 3$, thus $a \in S$, a contradiction. Thus, $V(G) - S$ in independent.

We now show that any vertex of S has at least a neighbor $V(G) - S$. Assume that a vertex $u \in S$ has not neighbor in $V(G) - S$. Let $e = uv \in E_3$. Then g defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $xy \in E_3 - \{uv\}$, 3 to v and 0 to any other vertex of G, is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction.

We now show that any vertex of S has at least two private neighbors $V(G) - S$. Assume that a vertex $x \in S$ has no private neighbor in $V(G) - S$. Let $e = uv \in E_3$. Then g defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_3 - \{uv\}$, 3 to v and 0 to any other vertex of G, is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction.

Next, assume that a vertex $x \in S$ has precisely one private neighbor in $V(G) - S$. Let $e = uv \in E_3$. Then g defined on $V(G)$ by assigning $f(e)$ to both x and y for any edge $e = xy \in E_3 - \{uv, ua\}$, 3 to v and 2 to a and 0 to any other vertex of G, is a DRDF for G of weight less than $2\gamma'_{dR}(G)$, a contradiction. We conclude that any vertex of S has at least two private neighbors in $V(G) - S$. Since $V(G) - S$ is independent, any vertex of S is a strong support vertex. Now, can see that $G \in \mathcal{E}$.

The conversely that $G \in \mathcal{E}$. Obviously that $\gamma_{dR}(G) \leqslant 6t$ and $\gamma'_{dR}(G) \leqslant 3t$. Assume f is $\gamma_{dR}(G)$-function, thus $w(f) = \gamma_{dR}(G)$. Let $u_i, v_i \in S$ and $z_i1, z_i2, ..., z_it$ and $z'_i1, z'_i2, ..., z'_it$ are leaves of u_i and v_i. Obviously $f(u_i) + \sum_{j=1}^{t_i} f(z_{ij}) \geqslant 3t$ and $f(v_i) + \sum_{j=1}^{t} f(z'_{ij}) \geqslant 3t$. Thus $f(u_i) + f(v_i) + \sum_{j=1}^{t_i} f(z_{ij}) + \sum_{j=1}^{t} f(z'_{ij}) \geqslant 6t \geqslant 2\gamma'_{dR}(G) \geqslant \gamma_{dR}(G)$.

□

Proposition 3.3. *The difference* $\gamma'_{dR}(T) - \gamma_{dR}(T)$ *can be arbitrarily large.*

Proof. For each integer $n \geqslant 2$, let G be a graph obtained from $K_{1,2n}$, by adding a perfect matching on the set of leaves of $K_{1,2n}$. Clearly, $\gamma_{dR}(G) \leqslant 3$ and $\gamma'_{dR}(G) \leqslant 2n + 1$. Thus $\gamma'_{dR}(G) - \gamma_{dR}(G) \leqslant 2(n-1)$. Let

$A_1, A_2, ..., A_n$, n, n triangle in $K_{1,2n}$. Then $f(A_i) \geqslant 2$ to each i. There is $1 \leqslant j \leqslant n$, such that $f(A_j) = 3$. Thus $\sum f(A_i) \leqslant 2n+1$. □

Proposition 3.4. *For any connected graph G of size $m \geqslant 2$ and $\Delta(G) \geqslant 2$, $\gamma'_{dR}(G) \leqslant 2m - 2\Delta(G) + 3$. Equality holds if and only if G is a star of order at least of three.*

Proof. Let v be a vertex of maximum degree $k = \Delta(G)$ and let $N(v) = \{v_1, v_2, ..., v_k\}$. Without loss of generality assume that $deg(v_1) \geqslant deg(v_i)$, $i = 2, 3, ..., k$. Define $f : E(G) \to \{0, 1, 2, 3\}$ by $f(vv_1) = 3$, $f(e) = 0$. If e is incident with v or v_1 and $f(e) = 2$ otherwise. It is easy to see that f is a EDRDF of G and so $\gamma'_{dR}(G) \leqslant 2m - 2(deg(v) + deg(v_1) - 2) + 3 \leqslant 2m - 2\Delta(G) + 3$. Assume that equality holds. Then $deg(v_1) = 1$. Consequently, G is a star of order at least three. The converse is obvious. □

Proposition 3.5. *Let G be a graph of size m, minimum degree $\delta \geqslant 1$ and maximum degree Δ. Then*

$$\gamma'_{dR}(G) \geqslant \tfrac{(2\delta+1)m}{(2\Delta-1)} - m$$

Proof. Assume that $g : E(G) \to \{0, 1, 2, 3\}$ is a $\gamma'_{dR}(G)$-function. Define $f : E(G) \to \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ by $f(e) = \frac{g(e)-1}{4}$ for each $e \in E(G)$. We have

$$\begin{aligned} \textstyle\sum_{e\in E} f(N_G[e]) &\geqslant \textstyle\sum_{e=uv\in E} \frac{g(N_G[e])+deg(u)+deg(v)-1}{4} \\ &\geqslant \tfrac{2m\delta}{4} + \textstyle\sum_{e=uv\in E} \frac{g(N_G[e])-1}{4} \\ &\geqslant \tfrac{2m\delta}{4} + \tfrac{m}{4} = \tfrac{(2\delta+1)m}{4}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \textstyle\sum_{e\in E} f(N_G[e]) &= \textstyle\sum_{e=uv\in E} (deg(u) + deg(v) - 1)f(e) \\ &\leqslant \textstyle\sum_{e\in E} (2\Delta - 1)f(e) \\ &= (2\Delta - 1)f(E(G)). \end{aligned}$$

BY (1) and (2), $f(E(G)) \geqslant \frac{(2\delta+1)m}{4(2\Delta-1)}$. Since $g(E(G)) = 4f(E(G)) - m$,

$$\gamma'_{dR}(G) = g(E(G)) \geqslant \tfrac{(2\delta+1)m}{(2\Delta-1)} - m,$$

as desired. □

Corollary 3.6. *Let G be a* r-regular *graph of order n. Then $\gamma'_{dR}(G) \geqslant \frac{nr}{(2r-1)}$.*

Theorem 3.7. *[16] Any line graph is claw-free.*

Theorem 3.8. *For any connected graph G of order $n \geqslant 4$ and size m, $\gamma'_{dR}(G) \leqslant \frac{5m}{4}$, equality holds if and only if $G = \frac{n}{5}P_5$.*

Proof. Let G be a connected graph of order $n \geqslant 4$. By theorem 2.2 and A, $\gamma'_{dR}(G) = \gamma_{dR}(L(G)) \leqslant \frac{5|V(L(G))|}{4} = \frac{5m}{4}$. Assume that equality holds. Then $\gamma_{dR}(L(G)) = \frac{5|V(L(G)|}{4}$ and by theorem A, $V(L(G))$ can be built from $\frac{n}{4}$copies of P_4 by adding a connected subgraph on the set of centers of $\frac{n}{4}P_4$. By Theorem 3.7, $L(G) = \frac{n}{4}P_4$. This implies that $G = \frac{n}{5}P_5$. The converse in obvious. □

Corollary 3.9. *For any tree T of order $n \geqslant 4$, $\gamma'_{dR}(T) \leqslant \frac{5n-5}{4}$, equality holds if and only if $T = \frac{n}{5}P_5$.*

We close this section with the following upper bound.

Theorem 3.10. *For any connected graph G of size m,*

$$\gamma'_{dR}(G) \leqslant 3\left(\tfrac{1+\ln(\Delta+\delta-1)}{2\delta-1}\right)m.$$

Proof. It is known in [7] that for any graph G of order n, $\gamma_{dR}(G) \leqslant 3\gamma(G)$. Now by observation 2.2,

$$\begin{array}{rcl} \gamma'_{dR}(G) & = & \gamma_{dR}(L(G)) \\ & \leqslant & 3\gamma(L(G)) \\ & \leqslant & 3(\frac{1+\ln(1+\delta(L(G))}{1+\delta(L(G))})n(L(G)) \\ & \leqslant & 3(\frac{1+\ln(\Delta+\delta-1)}{2\delta-1})m. \end{array}$$

□

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